Incremental Constraint-Based Elicitation of Multi-Attribute Utility Functions

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1 Introduction

In problem solving, the decision-theoretic framework is invoked when we wish to define a rich notion of preference over possible solutions. While decision theory provides a clean framwork for representing preference information, eliciting that information can often be difficult. When solutions are simply concrete outcomes, preferences can be represented directly by simply ranking the outcomes. But when solutions can have uncertain outcomes, this is not feasible and preferences are typically represented with a real-valued *utility function* over outcomes, which captures a decision-maker's atttitudes toward risk. Even in the case of complete certainty, preferences are often represented by a real-valued function over outcomes, called a *value function.*

Outcome space is typically defined in terms of a set of attributes, with an outcome being a complete assignment of values to all the attributes. For realistic decision-making problems the outcome space if usually too large to allow direct specification of utility and value functions, so assumptions are made that allow them to be specified in terms of component functions for each of the attributes. The assumptions concern independence of a user's preferences over values of one or more attributes as the values of other attributes are varied. Most work on elicitation has made the strong assumption of additive independence among the attributes. In this work we make the weaker assumption of simple utility independence among the attributes, which gives rise to a multi-linear form of utility function. Such a function is too complex to elicit directly, so instead we show how we can indirectly infer constraints on the utility function by eliciting pairwise preferences among solutions from the decision maker. We assume that the set of solutions is finite and that the decision maker is comfortable expressing pairwise preferences among some fraction of the solutions. For some problems it can be easier for a decision maker to express preference among solution alternatives than to introspect about the attributes describing each one. For example, in expressing preferences about movies most people can almost instantly express their preference among two films they have seen but may have difficulty describing preferences over attributes like director, leading actor, costume designer, etc. In fact, most people would not even recognize the names of costume designers.

2 Background

2.1 Multi-Attribute Utility Theory

In the most abstract form, multi-attribute utility theory is concerned with the evaluation of the consequences or outcomes of an agent's decisions or acts, where outcomes are characterized with complex sets of features called *attributes.* The attributes are henceforth denoted by $X_1, X_2, \ldots, X_n$, and outcomes are designated by $x = (x_1, \ldots, x_n)$, where x_i designates the value of attribute X_i. Abusing notation, we denote the set of values an attribute X_i can take simply by X_i, and thus the outcome space Ω is just the Carthesian product $X_1 \times X_2 \times \cdots \times X_n$. We will often talk about subsets Y of the set of attributes $X = \{X_1, X_2, \ldots, X_n\}$, and also refer to Y and their complements $Z = X - Y$ as attributes. With respect to such a pair (Y, Z), an outcome $x = (x_1, x_2, \ldots, x_n)$ can be written as (y, z). For example, if $n = 5$ and $Y = \{X_1, X_3\}$, then $y = (x_1, x_3)$ and $z = (x_2, x_4, x_5)$.

When acts have deterministic outcomes, in order to make decisions an agent need only express preferences among outcomes. The preference relation, henceforth denoted by $\succeq$, can often be captured by an order-preserving, real-valued *value function* v.

Given a preference order $\succeq$ over Ω, an attribute $Y \subset X$ is *preferentially independent* of its complement Z, or, for short, Y is PI, if the preference $\succeq$ over outcomes that are fixed in Z at some level does not depend on this level.

To infer the overall preference from such local preferences also seems straightforward; what we need is the direction agreement of the local preferences. Formally, we have the following proposition.

Proposition 1 *If both Y and its complement Z are PI, and $y' \succeq y'', z' \succeq z''$, then $(y', z') \succeq (y'', z'')$.*

When outcomes of acts are uncertain, an act is described in terms of a probability distribution over outcomes. To differentiate between certain and uncertain outcomes, we call uncertain outcomes *prospects*, and use *outcomes* to refer to outcomes with no uncertainty. Now the agent faces the more difficult task of ranking the prospects, instead of outcomes. The central result of utility theory is a representation theorem that proves the existence of a *utility function* $u : \Omega \to R$ such that preference order among prospects can be established based on the expectation of u over outcomes. If p is a prospect, then its expected utility is defined as:

$$u(p) := \sum_{x_i \in X_i} p(x_1, .., x_n) u(x_1, .., x_n) \qquad (1)$$

The key point here is that the utility function is defined over outcomes alone; the extension to prospects via expectation is a consequence of the axioms of probability and utility [7].

The generalization of preferential independence to the case of uncertainty is the concept of *utility independence* (UI). Given a preference order $\succeq$ over the prospects over Ω, an attribute $Y \subset X$ is *utility independent* of its complement Z, or, for short, Y is UI, if the preference $\succeq$ over prospects that are fixed in Z at some level does not depend on this level.

When applicable, utility independence gives us useful information about the form of the utility function. Towards this end, below we list a few relevant results, using [1] as our source.

Proposition 2 (Basic Decomposition) *If some attribute $Y \subset X$ is UI, then the utility function $u(x)$ must have the form:*

$$u(x) = u(y, z) = g(z) + h(z).u(y, z^+),$$

where $g(\cdot)$ and $h(\cdot) > 0$ depends only on z but not on y, and z^+ is some fixed value of z. The function $u(y, z^+)$, defined over variable y is called the subutility function *for the attribute Y, and is sometimes designated by $u_Y(y)$.*

From this result and the next few, we can see that utility independence plays just as fundamental a role in utility theory as does probabilistic independence in probability theory: it provides modularity and decomposition. When the UI of an attribute Y holds, the assessment of the utility function $u(x)$ is reduced to the assessment of three functions all of which have fewer arguments. This reduction of dimensionality is crucial, both analytically and practically.

If each attribute X_i is UI, the utility function can be expressed as a multi-linear function of sub-utility functions. That is there exist n sub-utility functions:

$$u_i : X_i \to R$$

Now if we set, for each subset $I \subset \{1, .., n\}$:

$$t_I(x) := \prod_{i \in I} u_i(x_i) \qquad (2)$$

then the utility function is just a weighted sum of the terms t_I:

$$u(x) = \sum_I k_I t_I(x) \qquad (3)$$

One problem of the multilinear utility function is that it requires the assesment of $O(2^n)$ scaling constants, in addition to the assesment of the subutility functions u_i. Stronger independence assumptions must hold in order for simpler forms of utility functions to be valid. So if we wish to elicit the utility function in terms of sub-utility funcitons and scaling constants we are faced with either assessing a huge number of scaling constants or making strong assumptions that may not hold. As an alterative, we propose to assume only the assumptions required for the multi-linear for to hold but to elicit the utility funcition by asking the decision maker for pairwise preferences between solutions and using these to infer constraints on the form of the utility function. The technique has the further benefit that the elicitation can be performed incrementally, inferring tighter and tighter constraints as we go.

2.2 Convex Polyhedral Cones

In general, the space of utility functions, as we will see soon, is a finite dimensional (real) vector space. A set of local constraints on the preference structure corresponds to a polyhedral cone in the space of utility functions, and vice versa. Given two outcomes x and y, the preference $x \succ y$ is equivalent to $u(x) > u(y)$, which, in term of equation (3), is:

$$\sum_I k_I t_I(x) > \sum_I k_I t_I(y) \qquad (4)$$

If we have preference on two prospects p and q, say $p \succ q$, then by equation (1) we have $u(p) > u(q)$ which is just:

$$\sum_I k_I t_I(p) > \sum_I k_I t_I(q) \qquad (5)$$

with $t_I(p) := \sum_x p(x) t_I(x)$. Now if we denote by $t(x)$ the vector of all $t_I(x)$, by $t(y)$ the vector of all

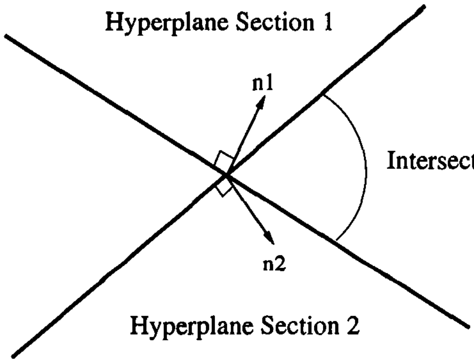

Figure 1: Hyperplane sections.

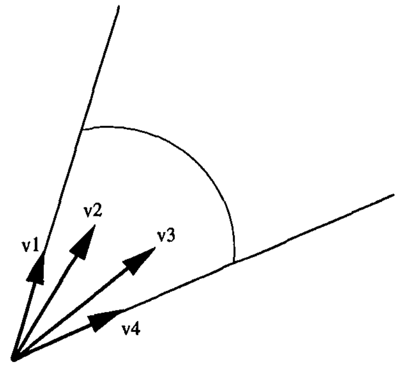

Figure 2: Convex cone generated by the set of vectors $\{v_1, v_2, v_3, v_4\}$

$t_I(y)$, and similarly for $t(p)$ and $t(q)$, then (4) and (5) become:

$$\langle k, t(x) \rangle > \langle k, t(y) \rangle \quad (6)$$

$$\langle k, t(p) \rangle > \langle k, t(q) \rangle \quad (7)$$

where $\langle *, * \rangle$ is the inner product of two vectors.

The above two inequalities give us two hyperplane sections on the space $\Re^d$ of points $k = (k_I)$, with normal vectors $n_1 = t(x) - t(y)$ and $n_2 = t(p) - t(q)$. Note that each point in $\Re^d$ is a list of coefficients of a utility function.

When we have several preferences over pairs of outcomes or prospects, the result is an intersection of a set of hyperplane sections, which is a polyhedral convex cone in $\Re^d$, by definition.

Definition 1 *The hyperplane with normal vector $n \in \Re^d$ is the set of points $x \in \Re^d$ satisfying the equation:*

$$\langle n, x \rangle = \sum_{i=1}^{d} n_i x_i = 0$$

We denote this hyperplane by $n^\perp$.

Definition 2 *The hyperplane section (as depicted in Figure 1) of a normal vector $n \in \Re^d$ is the set of points $x \in \Re^d$ lying above the hyperplane $n^\perp$:*

$$\langle n, x \rangle = \sum_{i=1}^{d} n_i x_i > 0$$

Denote this hyperplane section by n^+.

Definition 3 *A polyhedral convex cone is the intersection of a finite set of hyperplane sections $\{n_1^+, .., n_k^+\}$ in $\Re^d$.*

In the next proposition we see that a polyhedral cone can come about in in another way.

Proposition 3 *Each polyhedral convex cone is generated by a finite set of vectors $\{v_1, .., v_k\}$ in $\Re^d$:*

$$C(v_1, .., v_k) := \{x = \sum_{i=1}^{k} c_i v_i \in \Re^d \mid c_i \geq 0\}$$

An example of a polyhedral cone so generated is shown in Figure 2.

3 Induced Dominance

Suppose we have a finite set of n solutions from which we would like to select the most preferred solution. These solutions may involve uncertainty. For example they may be probabilistic plans. Each time, we can pick out from them a pair of solutions and ask the decision maker, which one he prefers. There is a trivial algorithm which after making $n-1$ such observations allows us to determine which solution is most preferred. We start by asking the decision maker for his preference among two arbitrary solutions. Then we ask for his preference between the most preferred and another solution. We continue, each time asking for his preference between the most preferred solution so far and one new solution. After $n-1$ comparisons we know the most preferred solution. This trivial algorithm is, of course, the worst we can do in terms of the number of questions.

In this section we will develop an algorithm that requires only $[log_2(n)]$ pairwise comparisons. We do this by using given preferences to induce new ones and then applying this idea to our main problem, that of finding optimal outcome(s) with minimal required information.

3.1 Pairwise Preferences

In practice we often have only a partial description of a preference structure. We would like to use that

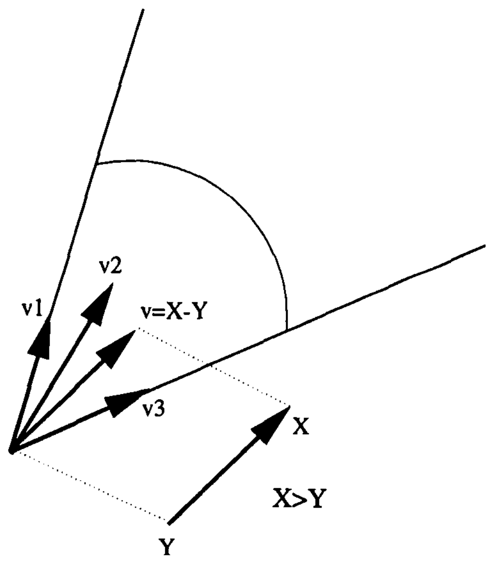

Figure 3: Induced cone from which we infer that $X \succ Y$

information to infer properties of the unspecified part of that structure.

Theorem 1 *Given a finite set of preferences* $\{X^{(i)} > Y^{(i)}\}$, *and a pair of outcomes* (X, Y). *Then we can infer that* $X \succ Y$ *if vector* $v = t(X) - t(Y)$ *belongs to the convex cone generated by vectors* $\{v_i = t(X^{(i)}) - t(Y^{(i)})\}$.

Figure 3 illustrates how we use this theorem to infer new preferences from given preferences.

In the picture, we have actually identified $t(X)$ with X, $t(Y)$ with Y,... This should not causes any ambiguities.

Proof.

If $v = t(X) - t(Y)$ belong to the generated cone then, by definition, there will be a set of nonnegative real numbers $\{c_i\}$ such that:

$$t(X) - t(Y) = \sum_i c_i(t(X^{(i)}) - t(Y^{(i)}))$$

Because in our preference, for each i, $X^{(i)} > Y^{(i)}$, so by (6) and (7) we have:

$$\langle k, t(X^{(i)})\rangle > \langle k, t(Y^{(i)})\rangle \quad (8)$$

and hence

$$\langle k, t(X^{(i)}) - t(Y^{(i)})\rangle = \langle k, t(X^{(i)})\rangle - \langle k, t(Y^{(i)})\rangle > 0 \quad (9)$$

and therefore

$$\langle k, t(X) - t(Y)\rangle = \sum_i c_i \langle k, t(X^{(i)}) - t(Y^{(i)})\rangle > 0 \quad (10)$$

$$\langle k, t(X)\rangle - \langle k, t(Y)\rangle = \langle k, t(X) - t(Y)\rangle > 0 \quad (11)$$

$$\langle k, t(X)\rangle > \langle k, t(Y)\rangle \quad (12)$$

Then by (6), X is preferred to Y in the preference structure.

Now we need an algorithm to check if a vector v belongs to a convex polyhedral cone C. This can be accomplished easily.

Call $n_1, n_2, .., n_k$ the normal vectors of the faces of C, each of which is pointing to the outside of C. Then v belongs to C if and only if $\langle n_i, v\rangle < 0$ for each $i = 1..k$. So all we need to know whether $v \in C$ or not, is to find the normal vectors of the faces of C, given the set of generators of C. This can be done incrementally as the set of generators expands during the elicitation process.

Algorithm 1

1. Compute the normal vectors $n_1, n_2, .., n_k$ from the generators of C.
2. Return $(x > y)$ if $\langle n_i, t(X) - t(Y)\rangle < 0$ for each $i = 1..k$.
3. Return $(x < y)$ if $\langle n_i, t(X) - t(Y)\rangle > 0$ for each $i = 1..k$.
4. Return nothing if otherwise.

Due to space requirement, we dont give here the details of how to incrementally compute the set of normal vectors. To give the reader an impression, we give the following picture to demonstrate how expansion of set of generators affect the set of faces of C.

In this section, we were concerned on inference on pairs. In the next one we will concentrate on how the partial description can reduce the set of possible optimal outcomes.

3.2 Exclusion of nonoptima with single preference

We have been given that $X > Y$ in the preference structure. We then want to know which ones in the set of outcomes $\{Y_1, .., Y_m\}$ are not optimal ones.

To solve this problem, we need a new notion, namely, the support cone at a vertex of a polytope.

Definition 4 *Given a convex polytope in the space* $\Re^d$ *and a vertex* P *of it. The smallest (intersection of all) cone(s) originating at* P *and containing the polytope is called the support cone at* P *of the polytope.*

We give a picture to illustrate this.

Now return to our problem. Call G the convex hull (a polytope) of the set of points $\{t(Y_1), t(Y_2), .., t(Y_m)\}$.

If any of the $\{t(Y_1), t(Y_2), .., t(Y_m)\}$ is not a vertex of G then it surely will not be the optimal one, because our utility function is linear hence optima can only be archived at the corner. So we can assume that each $t(Y_i)$ is a vertex of G.

Theorem 2 *If vector $t(X) - t(Y)$ belongs to the support cone sp_i of G at $t(Y_i)$ then Y_i is not the optimal outcome. In fact, it is less preferred than it's neighbours on the polytope.*

Proof.
Call the neighbours of vertex $t(Y_i)$ in the polytope G $\{t(Y_{i_1}, .., t(Y_{i_k})\}$. Then sp_i is generated by the vectors $\{t(Y_{i_1} - t(Y_i), .., t(Y_{i_k}) - t(Y_i)\}$. If $t(X) - t(Y)$ is in this cone, then we can write:

$$t(X) - t(Y) = \sum_j c_j(t(Y_{i_j} - t(Y_i))$$

where c_j are nonnegative real numbers. Hence we have:

$$\begin{aligned} 0 &< \langle k, t(X)\rangle - \langle k, t(Y)\rangle \\ &= \langle k, t(X) - t(Y)\rangle \\ &= \sum_j c_j \langle k, t(Y_{i_j} - t(Y_i))\rangle \\ &= \sum_j c_j \langle k, t(Y_{i_j})\rangle - \langle k, t(Y_i)\rangle \sum_j c_j \end{aligned}$$

So we get:

$$\langle k, t(Y_i)\rangle < \frac{1}{\sum_j c_j} \sum_j c_j \langle k, t(Y_{i_j})\rangle$$

This clearly give us:

$$\langle k, t(Y_i)\rangle < \langle k, t(Y_{i_{j_0}})\rangle$$

where $j_0 := \arg\max_j \langle k, t(Y_{i_j})\rangle$. So $t(Y_i)$ is not the optimal one.

Figure 4 illustrates this theorem.

In the picture, all vectors $YY', Y_1Y_1', Y_2Y_2', Y_3Y_3', Y_4Y_4'$ parallels to vector YX. Because $Y < X$, so this implies that Y, Y_1, Y_2, Y_3, Y_4 are less than $Y', Y_1', Y_2', Y_3', Y_4'$ accordingly. But $Y', Y_1', Y_2', Y_3', Y_4'$, in turn, are all smaller than max(A,B). So the alternatives $Y's$ are all smaller than max(A,B). This means that these alternatives are non-optimal ones.

`Algorithm 2`

1. Construct the support cones $\{sp_i\}$ from the set of outcomes $\{Y_i\}$.
   This can be incrementally done as in algorithm 1.
2. For each i, exclude Y_i if sp_i contains $t(X) - t(Y)$.
3. Return the remaining outcomes.

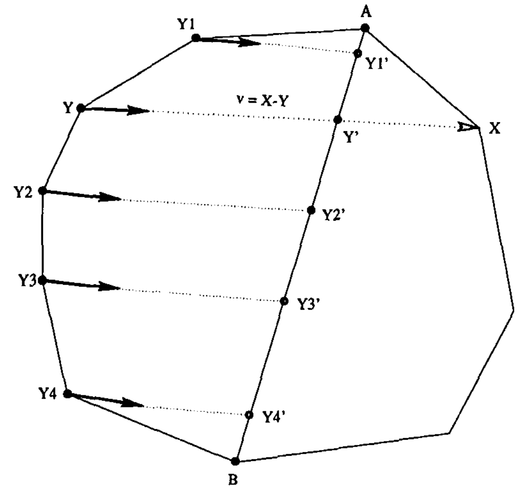

Figure 4: The preference $X \succ Y$ eliminates Y_1 through Y_4 as possible optimal solutions.

3.3 Exclusion of nonoptima with multiple preferences

We retreat the above problem in the case where we have more than one preference in hand. One of the approaches is to use each preference separately, so the algorithm is incremental. Each time, we pick an unused preference, use that preference as input for `algorithm 2` to exclude nonoptimal outcomes until all given preferences are used. Although, this is very effective, we still can do it better as the following theorem shows.

Theorem 3 *Given a set of outcomes $\{Y_1, .., Y_m\}$ and a set of preferences $\{X^{(i)} > Y^{(i)}, i = 1..k\}$. Call G the polytope with vertices $\{t(Y_1), .., t(Y_m)\}$, C the convex cone generate by vectors $\{X^{(i)} - Y^{(i)}, i = 1..k\}$. Then Y_i is not the optimal outcome if (and only if) the support cone sp_i intersects with the convex cone C nontrivially, ie. there exist at least a nonzero vector in $C \cap sp_i$. Where sp_i is the support cone at vertex $t(Y_i)$ of polytope G.*

Proof.
Call v a nonzero vector in $C \cap sp_i$. Because $v \in C$ so we can write

$$v = \sum_i c_i(X^{(i)} - Y^{(i)})$$

where c_i are nonnegative real numbers and not all equal zero. Hence we have

$$\langle k, v\rangle = \sum_i c_i \langle k, X^{(i)} - Y^{(i)}\rangle$$

Because for each i, $X^{(i)} > Y^{(i)}$, so $\langle k, X^{(i)} > Y^{(i)}\rangle > 0$, hence we have $\langle k, v\rangle > 0$. Now we know that $\langle k, v \rangle > 0$ and $v \in sp_i$. Similarly as in theorem 2, we then can show that Y_i is not the optimal outcome, where v is used instead of $t(X) - t(Y)$.

`Algorithm 3`

1. Construct the support cones $\{sp_i\}$ from the set of outcomes $\{Y_i\}$. This can be incrementally done as in algorithm 1.

2. Construct the cone C generated by set of vectors $C_v = \{t(X^{(i)}) - t(Y^{(i)})\}$.

3. For each i, exclude Y_i if sp_i intersects with C non trivially.

4. Return the remaining outcomes.

4 Optimal elicitation

In this section we solve the problem of optimally eliciting of optimal solutions in a controlled enviroment, i.e., where we can choose which pairs of alternatives to elicit the preference for. In this way, we will show that at most $[log_2(n)]$ controlled observations are needed in order to determine the best alternative in a set of n alternatives.

Parameterize each alternative A_i by the point $t(A_i) \in R^d$. We can form a convex polytope G from the set of alternatives by taking the convex hull of the set of points $\{t(A_i)\}$. Those points $t(A_k)$ that are not vertices of G correspond to non-optimal alternatives A_k, so we can safely remove all such alternatives from our set.

From the previous section, we know that if x is prefrred to y in the preference structure then for any i such that $t(x) - t(y)$ belongs to the support cone at vertex $t(A_i)$ of polytope G, A_i is a non-optimal alternative. If the set of alternatives are distributed uniformly randomly, then about half of the alternatives are eliminated by the expression of any preference $x \succ y$.

As the elicitation process continues, the the set of remaining alternatives reduces smaller and smaller. At any point in time, we still can choose a pair of outcomes x, y such that either $x \succ y$ or $y \succ x$ will help us to eliminate half of the remaining alternatives. This can be shown easily by the diagram in Figure 5.

If $X \succ Y$ then all the alternatives lying on the *left* side of the dashed line are non-optimal. But if $X \prec Y$ then it will be the other way around: all the alternatives lying on the *right* side of the dashed line are non-optimal.

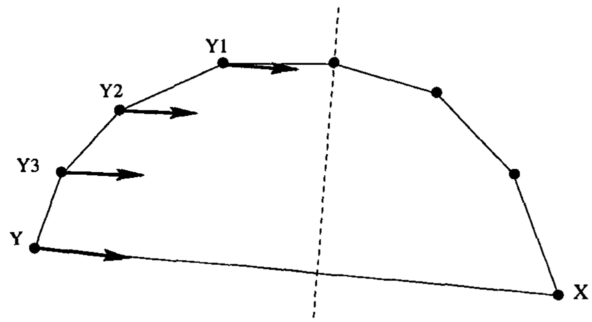

Figure 5: Given $X \succ Y$, all alternatives to the left of the dashed line are sub-optimal.

5 Empirical Results and Related Work

Zionts [8] and Koksalan [2] have addressed the same problem we address in this paper but under the assumption that the underlying utility function is linearly additive. Korhonen et. al. [6] later generalized their results to apply to quasi-concave utility functions, i.e., monotonically increasing with a convex part and a concave part. Quasi-concave utility functions are a subset of the multi-linear utility functions we consider. Koksalan [3; 5] later attacked the same problem as Korhonen et. al. but by using the concept of local dominance. The technique determines an set of locally optimal solutions and then compares these to determine a global optimum. He subsequently extended his method to deal with general monotone utility functions [4]. Again, monotone utility functions are a subset of multi-linear utility functions.

We compare our `algorithm` 2 to those discussed above, using the data shown in the paper by Koksalan [4]. Koksalan evaluated his technique and those in [6] and [5] on three problems of increasing size. In each problem a finite number of candidate solutions was randomly chosen from a continuous solution space. Each technique was allowed to ask for pairwise comparisons between any two points in the continuous space. In contrast, our method only elicits pariwise comparisons between actual candidate solutions, thus making the elicitation problem more realistic and more difficult.

The two tables below show the performance of each method for a solution space defined by three attributes. The first row of each table shows the number of candidate solutions; the remaining rows show the number of questions asked of the decision maker in order to determine the optimal solution using each method. The first table shows the results for a concave utilty function. For this restricted type of function the other methods perform equally as well as ours or outperform ours by a factor of about two.

The techniques in [6] and [5] are not applicable to the more general class of monotone utility functions. The second table compares our method to that of Koksalen [4] for such a function. As we can see, despite the restriction on the types of questions our approach can ask, it outperforms Koksalen's method by as much as a factor of three, and the number of questions asked increases only little as a function of problem size. Our technique performs well on large problems because the candidate solutions tend to be more uniformly distributed throughout the solution space. This allows each of the first few questions to eliminate about half of the remaining candidates. Our technique gains computational leverage by working with the space of coefficients of the multi-linear utility function, while the other approaches discussed work only with the solution space.

Concave Tchebycheff Utility function

problem size:	100.0	200.0	300.0
Korhonen84	24.8	35.0	36.2
Koksalan92	10.9	15.0	17.0
Koksalan95	17.2	14.2	16.8
Our method	24.0	32.0	36.0

Monotone Utility Function

problem size:	100.0	200.0	300.0
Koksalan95	36.9	96.3	107.4
Our method	24.0	32.0	36.0

To determine the incremental performance of our algorithm, we ran some preliminary experiments with a monotone utility function. We randomly generated a set of 300 candidate solutions on which we ran our algorithm. One of the strong points of our algorithm is its first step, where solutions lying inside the convex hull of the set of all solutions in $2^d - 1$-dimension space are removed. In this step it eliminated 237 of the 300 candidate solutions. After this first step, we know that there are 63 solutions remaining so at most 64 questions will be required to determine the best solution. In contrast, Koksalan 95 needed 133 questions in order to solve the problem, as shown in the first row of the second table. Although our algorithm 3 is potentialy better than algorithm 2, we have not yet had a chance to empirically evaluate it.

Acknowledgements

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